## Supplementary Material

**Sketchy Proof of Theorem 1.**

Similar to the treatment in (Barber & Candès, 2015), we only need to prove that the knockoff statistics $W_j$ satisfy the following two properties:

- *sufficiency property*: $W = f([\delta_0, A_{ko}]^T[\delta_0, A_{ko}], [\delta_0, A_{ko}]^T Y)$, which indicates $W$ depends only on $[\delta_0, A_{ko}]^T[\delta_0, A_{ko}]$ and $[\delta_0, A_{ko}]^T Y$.
- *antisymmetry property*: Swapping $A_j$ and $\tilde{A}_j$ has the effect of switching the sign of $W_j$.

The second property is obvious because $W_j$ is constructed using entering time difference. Now we go to prove the first property.

For ISS and LBI, the whole path is only determined by

$$A_{ko}^T(Y - \delta_0\theta - A_{ko}\gamma_{ko}) = A_{ko}^T Y - A_{ko}^T[\delta_0, A_{ko}][\theta^T, \gamma_{ko}^T]^T),$$
$$\delta_0^T(Y - \delta_0\theta - A_{ko}\gamma_{ko}) = \delta_0^T Y - \delta_0^T[\delta_0, A_{ko}][\theta^T, \gamma_{ko}^T]^T),$$

which is only based on $[\delta_0, A_{ko}]^T[\delta_0, A_{ko}]$ and $[\delta_0, A_{ko}]^T Y$, so is the entering time $Z_j$

The same reasoning holds for LASSO since

$$\min_{\theta,\gamma} \frac{1}{2}\|Y - [\delta_0, A_{ko}][\theta^T, \gamma_{ko}^T]^T\|_2^2 + \lambda\|\gamma_{ko}\|_1$$

is equivalent to

$$\min_{\theta,\gamma} \quad \tfrac{1}{2}(\|Y\|_2^2 + [\theta^T, \gamma_{ko}^T][\delta_0, A_{ko}]^T[\delta_0, A_{ko}][\theta^T, \gamma_{ko}^T]^T - 2[\theta^T, \gamma_{ko}^T][\delta_0, A_{ko}]^T Y) + \lambda\|\gamma_{ko}\|_1$$

So the entire path is determined by $[\delta_0, A_{ko}]^T[\delta_0, A_{ko}]$ and $[\delta_0, A_{ko}]^T Y$.

**Proof of Theorem 2.**

Suppose $\tilde{X}$ is the knockoff statistics for (10), then it satisfies

$$\tilde{X}^T\tilde{X} = X^T X, X^T\tilde{X} = X^T X - \mathrm{diag}(s). \qquad (12)$$

Let $B = A + U_2(\tilde{X} - X)$, then $\tilde{X} = U_2^T B$ and it can be verified

$$B^T B = A^T A, A^T B = A^T A - \mathrm{diag}(s), \delta_0^T B = \delta_0^T A$$

which means $B$ is a valid knockoff feature matrix for (3).

On the reverse, let $\tilde{A}$ be knockoff features for (3), it is also easy to verify $\tilde{X} = U_2^T\tilde{A}$ satisfies condition (12). This establishes an injection between $\tilde{X}$ and $\tilde{A}$.

The equivalence of knockoff statistics comes from the equivalence of solution paths in both approaches. To see this, (4b) actually means $\hat{\theta} = (\delta_0^T\delta_0)^\dagger\delta_0^T(Y - A_{ko}\gamma_{ko})$, plugging $\hat{\theta}$ in (4a), we get

$$\begin{aligned}\frac{dp}{dt} &= A_{ko}^T(Y - \delta_0\hat{\theta} - A_{ko}\gamma_{ko}) \\ &= A_{ko}^T(U_2 U_2^T(Y - A_{ko}\gamma_{ko})) \\ &= (U_2^T A_{ko})^T(U_2^T Y - U_2^T A_{ko}\gamma_{ko})\end{aligned}$$

This is equivalent to the ISS for the second procedure model (9) in Remark 1. So in both approaches, the two ISS solution paths are identical.

The same reasoning holds for LASSO, the derivative of (5) w.r.t. $\theta$ is zero at the optimal estimator which means

$$0 = \delta_0^T(Y - \delta_0\hat{\theta} - A_{ko}\gamma_{ko})$$

this is actually (4b). So plugging $\hat{\theta}$ in (5), we get

$$\begin{aligned}&\|Y - \delta_0\theta - A_{ko}\gamma_{ko}\|_2^2 \\ =\ &\|(I - \delta_0(\delta_0^T\delta_0)^\dagger\delta_0^T)^T(Y - A_{ko}\gamma_{ko})\|_2^2 \\ =\ &\|U_2 U_2^T(Y - A_{ko}\gamma_{ko})\|_2^2 \\ =\ &\|U_2^T Y - U_2^T A_{ko}\gamma_{ko})\|_2^2.\end{aligned}$$

This is in fact the $l_2$ loss for the second procedure in Remark 1.

Finally identical paths lead to the same knockoff statistics which ends the proof.