A Mathematical Introduction to Data Science **Apr. 12, 2025**

Homework 9. Combinatorial Hodge Theory

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The problem below marked by * is optional with bonus credits.

1. *HodgeRank:* Download the HodgeRank matlab codes and unzip:

   `https://yao-lab.github.io/publications/BatchHodge.zip`

   which contains two subfolders.

   - `./data/`: file `incomp.mat` contains a 1668-by-2 matrix, collecting 1668 pairwise comparisons among 16 video items, with the first column index preferred to the second ones;
   - `./code/`: file `Hodgerank.m` is the Hodge decomposition of such pairwise comparison data.

   Run the following command

   ```
   >> load data/incomp.mat
   >> cd code
   >> [score,totalIncon,harmIncon] = Hodgerank(incomp)
   ```

   You will return with global ranking scores (generalized Borda count) in `score`, a 16-by-4 matrix as scores of 16 videos in 4 models:

   ```
   model1: Uniform noise model, Y_hat(i,j) = 2 pij -1
   model2: Bradley-Terry, Y_hat(i,j) = log(abs(pij+eps))-log(abs(1-pij-eps))
   model3: Thurstone-Mosteller, Y_hat(i,j) ~ norminv(abs(1-pij-eps));
           model4: Arcsin, Y_hat4(i,j) = asin(2*pij-1)
   ```

   and two inconsistency measurements (total inconsistency `totalIncon` = harmonic inconsistency `harmIncon` + triangular inconsistency). The following ratio:

   ```
   >> harmIncon/totalIncon
   ```

   measures the percentage of harmonic inconsistency in the total inconsistency (residue).

   Moreover, *can you compute the HodgeRank for the weblink data?* For example, the following dataset contains Chinese (mainland) University Weblink during 12/2001-1/2002,

   `https://github.com/yao-lab/yao-lab.github.io/blob/master/data/univ_cn.mat`

   compute the HodgeRank scores and compare them against PageRank and HITs etc.

Reference:

- Xiaoye Jiang, Lek-Heng Lim, Yuan Yao and Yinyu Ye. *Statistical Ranking and Combinatorial Hodge Theory.* Mathematical Programming, Volume 127, Number 1, Pages 203-244, 2011.
- Qianqian Xu, Qingming Huang, Tingting Jiang, Bowei Yan, Weisi Lin and Yuan Yao, *HodgeRank on Random Graphs for Subjective Video Quality Assessment*, IEEE Transaction on Multimedia, vol. 14, no. 3, pp. 844-857, 2012.

2. *Hodge Decomposition in Linear Algebra.* For inner product spaces X, Y, and Z, consider

$$X \xrightarrow{A} Y \xrightarrow{B} Z.$$

and $\Delta = AA^T + B^T B : Y \to Y$ where A^T (B^T) is the adjoint of A (B) such that $\langle Ax, y\rangle = \langle x, A^T y\rangle$ ($\langle y, B^T z\rangle = \langle By, z\rangle$), respectively. Show that if the following composition vanishes,

$$B \circ A = 0,$$

then $\ker(\Delta) = \ker(A^T) \cap \ker(B)$ and the following *orthogonal* decomposition holds

$$Y = \mathrm{image}(A) + \ker(\Delta) + \mathrm{image}(B^T).$$

3. **Hodge Decomposition of the Prisoner's Dilemma Game*: Consider the normal form game of the Prisoner's Dilemma, whose row and column players can play C (cooperate) or D (defect) and receive the payoffs, respectively (as in the table). Show that *the Hodge Decomposition of its game flow is a potential game.*

- Prisoner's Dilemma:

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- Putin's War against Nato:

	Expand	Stop
Attack	−10, −10	2, 1
Quit	−20, 5	1, 2

Reference: Ozan Candogan, Ishai Menache, Asuman Ozdaglar, and Pablo A. Parrilo. *Flows and Decompositions of Games: Harmonic and Potential Games.* Mathematics of Operations Research, 36(3): 474 - 503, 2011.