A Mathematical Introduction to Data Science

Apr. 12, 2025

Homework 9. Combinatorial Hodge Theory

Instructor: Yuan Yao Due: 1 weeks later

The problem below marked by * is optional with bonus credits.

1. HodgeRank: Download the HodgeRank matlab codes and unzip:

https://yao-lab.github.io/publications/BatchHodge.zip which contains two subfolders.

- ./data/: file incomp.mat contains a 1668-by-2 matrix, collecting 1668 pairwise comparisons among 16 video items, with the first column index preferred to the second ones;
- ./code/: file Hodgerank.m is the Hodge decomposition of such pairwise comparison data.

Run the following command

- >> load data/incomp.mat
- >> cd code
- >> [score,totalIncon,harmIncon] = Hodgerank(incomp)

You will return with global ranking scores (generalized Borda count) in score, a 16-by-4 matrix as scores of 16 videos in 4 models:

and two inconsistency measurements (total inconsistency totalIncon = harmonic inconsistency harmIncon + triangular inconsistency). The following ratio:

>> harmIncon/totalIncon

measures the percentage of harmonic inconsistency in the total inconsistency (residue).

Moreover, can you compute the HodgeRank for the weblink data? For example, the following dataset contains Chinese (mainland) University Weblink during 12/2001-1/2002,

https://github.com/yao-lab/yao-lab.github.io/blob/master/data/univ_cn.mat compute the HodgeRank scores and compare them against PageRank and HITs etc.

Reference:

- Xiaoye Jiang, Lek-Heng Lim, Yuan Yao and Yinyu Ye. Statistical Ranking and Combinatorial Hodge Theory. Mathematical Programming, Volume 127, Number 1, Pages 203-244, 2011.
- Qianqian Xu, Qingming Huang, Tingting Jiang, Bowei Yan, Weisi Lin and Yuan Yao, HodgeRank on Random Graphs for Subjective Video Quality Assessment, IEEE Transaction on Multimedia, vol. 14, no. 3, pp. 844-857, 2012.
- 2. Hodge Decomposition in Linear Algebra. For inner product spaces X, Y, X and Z, consider

$$X \xrightarrow{A} Y \xrightarrow{B} Z$$
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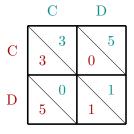
and $\Delta = AA^T + B^TB : Y \to Y$ where $A^T(B^T)$ is the adjoint of A(B) such that $\langle Ax, y \rangle = \langle x, A^Ty \rangle$ ($\langle y, B^Tz \rangle = \langle By, z \rangle$), respectively. Show that if the following composition vanishes,

$$B \circ A = 0$$
,

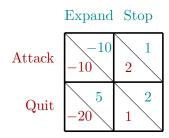
then $\ker(\Delta) = \ker(A^T) \cap \ker(B)$ and the following *orthogonal* decomposition holds

$$Y = \operatorname{image}(A) + \ker(\Delta) + \operatorname{image}(B^T).$$

- 3. *Hodge Decomposition of the Prisoner's Dilemma Game: Consider the normal form game of the Prisoner's Dilemma, whose row and column players can play C (cooperate) or D (defect) and receive the payoffs, respectively (as in the table). Show that the Hodge Decomposition of its game flow is a potential game.
 - Prisoner's Dilemma:



• Putin's War against Nato:



Reference: Ozan Candogan, Ishai Menache, Asuman Ozdaglar, and Pablo A. Parrilo. Flows and Decompositions of Games: Harmonic and Potential Games. Mathematics of Operations Research, 36(3): 474 - 503, 2011.