

Homework 9. Combinatorial Hodge Theory

*Instructor: Yuan Yao**Due: 1.5 weeks later*

The problem below marked by * is optional with bonus credits.

1. *HodgeRank*: Download the HodgeRank matlab codes and unzip:

<https://yao-lab.github.io/publications/BatchHodge.zip>

which contains two subfolders.

- `./data/`: file `incomp.mat` contains a 1668-by-2 matrix, collecting 1668 pairwise comparisons among 16 video items, with the first column index preferred to the second ones;
- `./code/`: file `Hodgerank.m` is the Hodge decomposition of such pairwise comparison data.

Run the following command

```
>> load data/incomp.mat
>> cd code
>> [score,totalIncon,harmIncon] = Hodgerank(incomp)
```

You will return with global ranking scores (generalized Borda count) in `score`, a 16-by-4 matrix as scores of 16 videos in 4 models:

```
model1: Uniform noise model,  $Y_{\hat{}}(i,j) = 2 p_{ij} - 1$ 
model2: Bradley-Terry,  $Y_{\hat{}}(i,j) = \log(\text{abs}(p_{ij}+\text{eps})) - \log(\text{abs}(1-p_{ij}-\text{eps}))$ 
model3: Thurstone-Mosteller,  $Y_{\hat{}}(i,j) \sim \text{norminv}(\text{abs}(1-p_{ij}-\text{eps}))$ ;
model4: Arcsin,  $Y_{\hat{}}(i,j) = \text{asin}(2*p_{ij}-1)$ 
```

and two inconsistency measurements (total inconsistency `totalIncon` = harmonic inconsistency `harmIncon` + triangular inconsistency). The following ratio:

```
>> harmIncon/totalIncon
```

measures the percentage of harmonic inconsistency in the total inconsistency (residue).

Moreover, *can you compute the HodgeRank for the weblink data?* For example, the following dataset contains Chinese (mainland) University Weblink during 12/2001-1/2002,

https://github.com/yao-lab/yao-lab.github.io/blob/master/data/univ_cn.mat

compute the HodgeRank scores and compare them against PageRank and HITs etc.

Reference:

- Xiaoye Jiang, Lek-Heng Lim, Yuan Yao and Yinyu Ye. *Statistical Ranking and Combinatorial Hodge Theory*. Mathematical Programming, Volume 127, Number 1, Pages 203-244, 2011.
- Qianqian Xu, Qingming Huang, Tingting Jiang, Bowei Yan, Weisi Lin and Yuan Yao, *HodgeRank on Random Graphs for Subjective Video Quality Assessment*, IEEE Transaction on Multimedia, vol. 14, no. 3, pp. 844-857, 2012.

2. *Hodge Decomposition in Linear Algebra*. For inner product spaces X , Y , and Z , consider

$$X \xrightarrow{A} Y \xrightarrow{B} Z.$$

and $\Delta = AA^T + B^TB : Y \rightarrow Y$ where A^T (B^T) is the adjoint of A (B) such that $\langle Ax, y \rangle = \langle x, A^T y \rangle$ ($\langle y, B^T z \rangle = \langle By, z \rangle$), respectively. Show that if the following composition vanishes,

$$B \circ A = 0,$$

then $\ker(\Delta) = \ker(A^T) \cap \ker(B)$ and the following *orthogonal* decomposition holds

$$Y = \text{image}(A) + \ker(\Delta) + \text{image}(B^T).$$

3. **Hodge Decomposition of the Prisoner's Dilemma Game*: Consider the normal form game of the Prisoner's Dilemma, whose row and column players can play C (cooperate) or D (defect) and receive the payoffs, respectively (as in the table). Show that *the Hodge Decomposition of its game flow is a potential game*.

- Prisoner's Dilemma:

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

- Putin's War against Nato:

	Expand	Stop
Attack	-10, -10	1, 2
Quit	5, -20	2, 1

Reference: Ozan Candogan, Ishai Menache, Asuman Ozdaglar, and Pablo A. Parrilo. *Flows and Decompositions of Games: Harmonic and Potential Games*. Mathematics of Operations Research, 36(3): 474 - 503, 2011.