Theory and Application of Copula

CCER

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Econometrics
（计量经济学）

- What is Econometrics？

  Combination of economics, mathematics and statistics.

- What is the focus of econometrics?

  Estimating Various Conditional Moments

  $E(Y|X)$, $Var(Y|X)$, $P(Y<=q|X)$

  Estimating Various Structural Models from Economic Theory.

My Focuses

- Modelling Time Series
  Nonstationary Data and Nonlinear Models
- Monitoring Structural Change
- Applying Copula Functions

A Heuristic Example

How to construct a two-variate joint distribution function whose marginals are respectively but not jointly normally distributed?

$$X \sim F(x),\ Y \sim G(y),\ (X,Y) \sim H(x,y)$$

$$H(x,y) = H(F^{-1}(F(x)), G^{-1}(G(y))) = C(u,v) \circ (F(x), G(y))$$

$$where\ C(u,v) = H(F^{-1}(u), G^{-1}(v))$$

Sklar Theorem (1959)

Let H be a n-dimensional distbution function with margins $F_1,...,F_n$. Then there exists an n-copula C such that for all $x \in R^n$,

$$H(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n))$$

C is unique if $F_1,...,F_n$ are all continuous.

Conversely, if is C a n-copula and $F_1,...,F_n$ are distribution functions, then H defined above is an n-dimensional distribution function with margins $F_1,...,F_n$.

Copula

- Definition (see Nelson (2006) definition 2.10.6)

$C\ (u_1,...,u_n)$ is a distribuiton function whose marginals are all uniformly distributed.

- Why we need copula?
- Discrete copula?

Copula Constructing

- Inversion Method
- Geometric Method
- Algebraic Method
- Other Methods?

Archimedean Copula Families

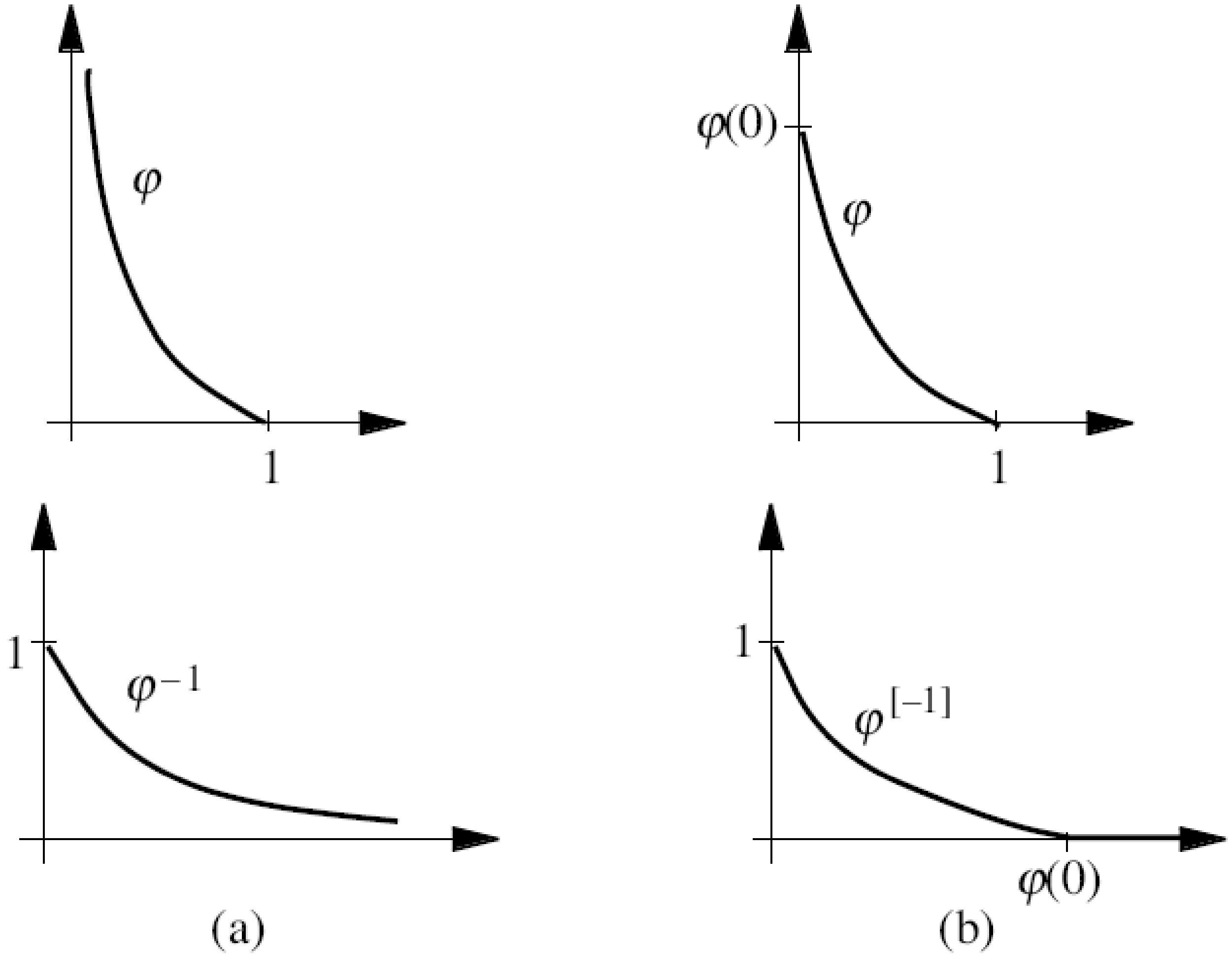
Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty)$ such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of $\varphi,$ that is,

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1} & 0\le \mathrm{t}\le \varphi(0)\\ 0 & \varphi(0)\le t\le \infty\end{cases}.$$

Then the function $C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v))$ is a copula iff φ is convex.

where $C(u,v)$ φ are called as an Archimedean copula and corresponding genarator respectively.

Archimedean Copula Families



Strict (a) and non-strict (b) generators and inverses

Archimedean Copula Families

Table 4.1. One-parameter

(4.2.#)	$C_\theta(u,v)$	$\varphi_\theta(t)$
1	$\left[\max\left(u^{-\theta}+v^{-\theta}-1,0\right)\right]^{-1/\theta}$	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$
2	$\max\left(1-\left[(1-u)^\theta+(1-v)^\theta\right]^{1/\theta},0\right)$	$(1-t)^\theta$
3	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\ln\frac{1-\theta(1-t)}{t}$
4	$\exp\left(-\left[(-\ln u)^\theta+(-\ln v)^\theta\right]^{1/\theta}\right)$	$(-\ln t)^\theta$
5	$-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$

Archimedean Copula Families

6	$1-\left[(1-u)^{\theta}+(1-v)^{\theta}-(1-u)^{\theta}(1-v)^{\theta}\right]^{1/\theta}$	$-\ln\left[1-(1-t)^{\theta}\right]$
7	$\max\left(\theta uv+(1-\theta)(u+v-1),0\right)$	$-\ln\left[\theta t+(1-\theta)\right]$
8	$\max\left(\dfrac{\theta^{2}uv-(1-u)(1-v)}{\theta^{2}-(\theta-1)^{2}(1-u)(1-v)},0\right)$	$\dfrac{1-t}{1+(\theta-1)t}$
9	$uv\exp\left(-\theta\ln u\ln v\right)$	$\ln\left(1-\theta\ln t\right)$
10	$uv\Big/\left[1+(1-u^{\theta})(1-v^{\theta})\right]^{1/\theta}$	$\ln\left(2t^{-\theta}-1\right)$
11	$\left[\max\left(u^{\theta}v^{\theta}-2(1-u^{\theta})(1-v^{\theta}),0\right)\right]^{1/\theta}$	$\ln\left(2-t^{\theta}\right)$

Archimedean Copula Families

12	$\left(1+\left[(u^{-1}-1)^{\theta}+(v^{-1}-1)^{\theta}\right]^{1/\theta}\right)^{-1}$	$\left(\frac{1}{t}-1\right)^{\theta}$
13	$\exp\left(1-\left[(1-\ln u)^{\theta}+(1-\ln v)^{\theta}-1\right]^{1/\theta}\right)$	$(1-\ln t)^{\theta}-1$
14	$\left(1+\left[(u^{-1/\theta}-1)^{\theta}+(v^{-1/\theta}-1)^{\theta}\right]^{1/\theta}\right)^{-\theta}$	$(t^{-1/\theta}-1)^{\theta}$
15	$\left\{\max\left(1-\left[(1-u^{1/\theta})^{\theta}+(1-v^{1/\theta})^{\theta}\right]^{1/\theta},0\right)\right\}^{\theta}$	$\left(1-t^{1/\theta}\right)^{\theta}$
16	$\frac{1}{2}\left(S+\sqrt{S^2+4\theta}\right),\ S=u+v-1-\theta\left(\frac{1}{u}+\frac{1}{v}-1\right)$	$\left(\frac{\theta}{t}+1\right)(1-t)$
17	$\left(1+\frac{[(1+u)^{-\theta}-1][(1+v)^{-\theta}-1]}{2^{-\theta}-1}\right)^{-1/\theta}-1$	$-\ln\frac{(1+t)^{-\theta}-1}{2^{-\theta}-1}$

Archimedean Copula Families

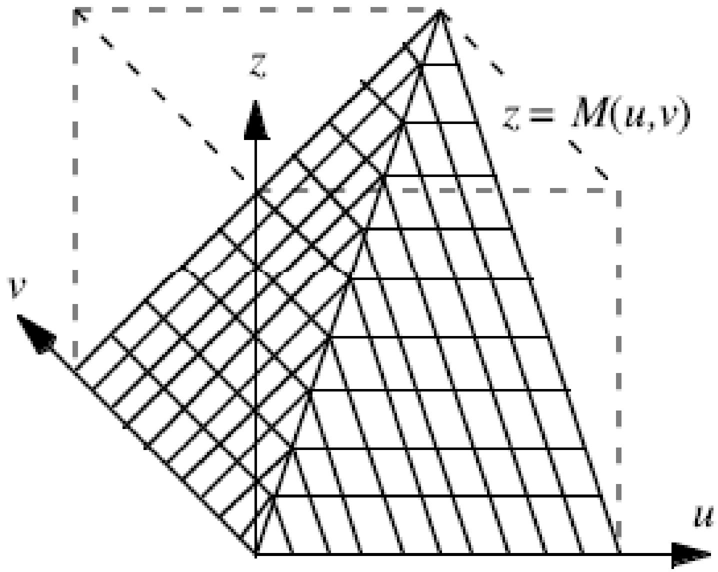
18	$\max\left(1+\theta\big/\ln\left[e^{\theta/(u-1)}+e^{\theta/(v-1)}\right],0\right)$	$e^{\theta/(t-1)}$
19	$\theta\big/\ln\left(e^{\theta/u}+e^{\theta/v}-e^{\theta}\right)$	$e^{\theta/t}-e^{\theta}$
20	$\left[\ln\left(\exp(u^{-\theta})+\exp(v^{-\theta})-e\right)\right]^{-1/\theta}$	$\exp\left(t^{-\theta}\right)-e$
21	$1-(1-\{\max([1-(1-u)^{\theta}]^{1/\theta}+[1-(1-v)^{\theta}]^{1/\theta}-1,0)\}^{\theta})^{1/\theta}$	$1-\left[1-(1-t)^{\theta}\right]^{1/\theta}$
22	$\max\left(\left[1-(1-u^{\theta})\sqrt{1-(1-v^{\theta})^{2}}-(1-v^{\theta})\sqrt{1-(1-u^{\theta})^{2}}\right]^{1/\theta},0\right)$	$\arcsin\left(1-t^{\theta}\right)$

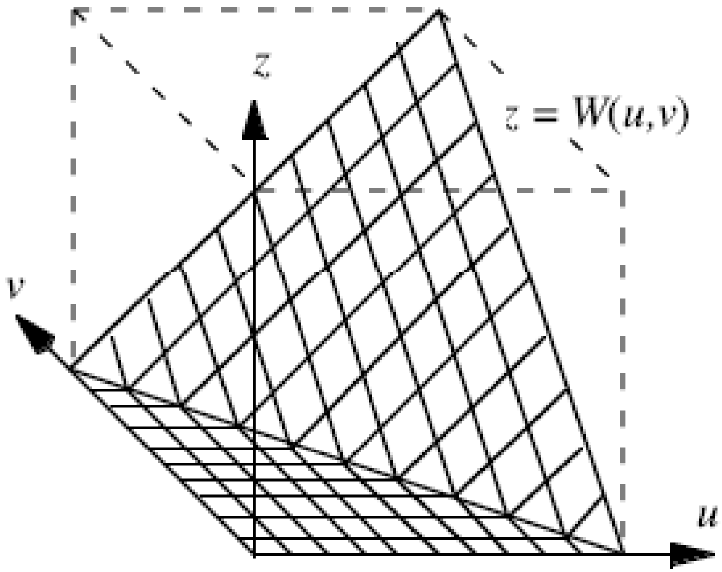
Archimedean Copula Families

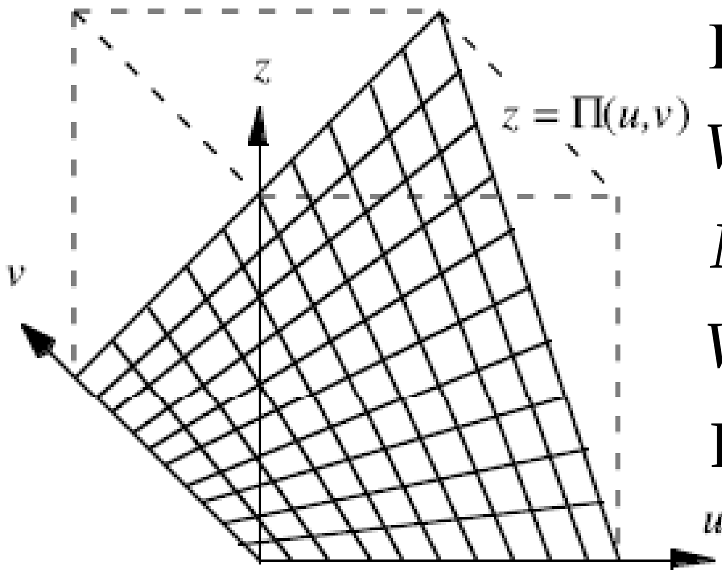
Families of Archimedean Copulas

$\theta \in$	Strict	Limiting and Special Cases	(4.2.#)
$[-1,\infty)\backslash\{0\}$	$\theta \geq 0$	$C_{-1} = W,\ C_0 = \Pi,\ C_1 = \dfrac{\Pi}{\Sigma - \Pi},\ C_\infty = M$	1
$[1,\infty)$	no	$C_1 = W,\ C_\infty = M$	2
$[-1,1)$	yes	$C_0 = \Pi,\ C_1 = \dfrac{\Pi}{\Sigma - \Pi}$	3
$[1,\infty)$	yes	$C_1 = \Pi,\ C_\infty = M$	4
$(-\infty,\infty)\backslash\{0\}$	yes	$C_{-\infty} = W,\ C_0 = \Pi,\ C_\infty = M$	5

Archimedean Copula Families







$$\Pi(u,v) = uv$$

$$W(u,v) = \max(u+v-1,0)$$

$$M(u,v) = \min(u,v)$$

$$W(u,v) \le C(u,v) \le M(u,v)$$

Fréchet-Hoeffding bounds

Archimedean Copula Families

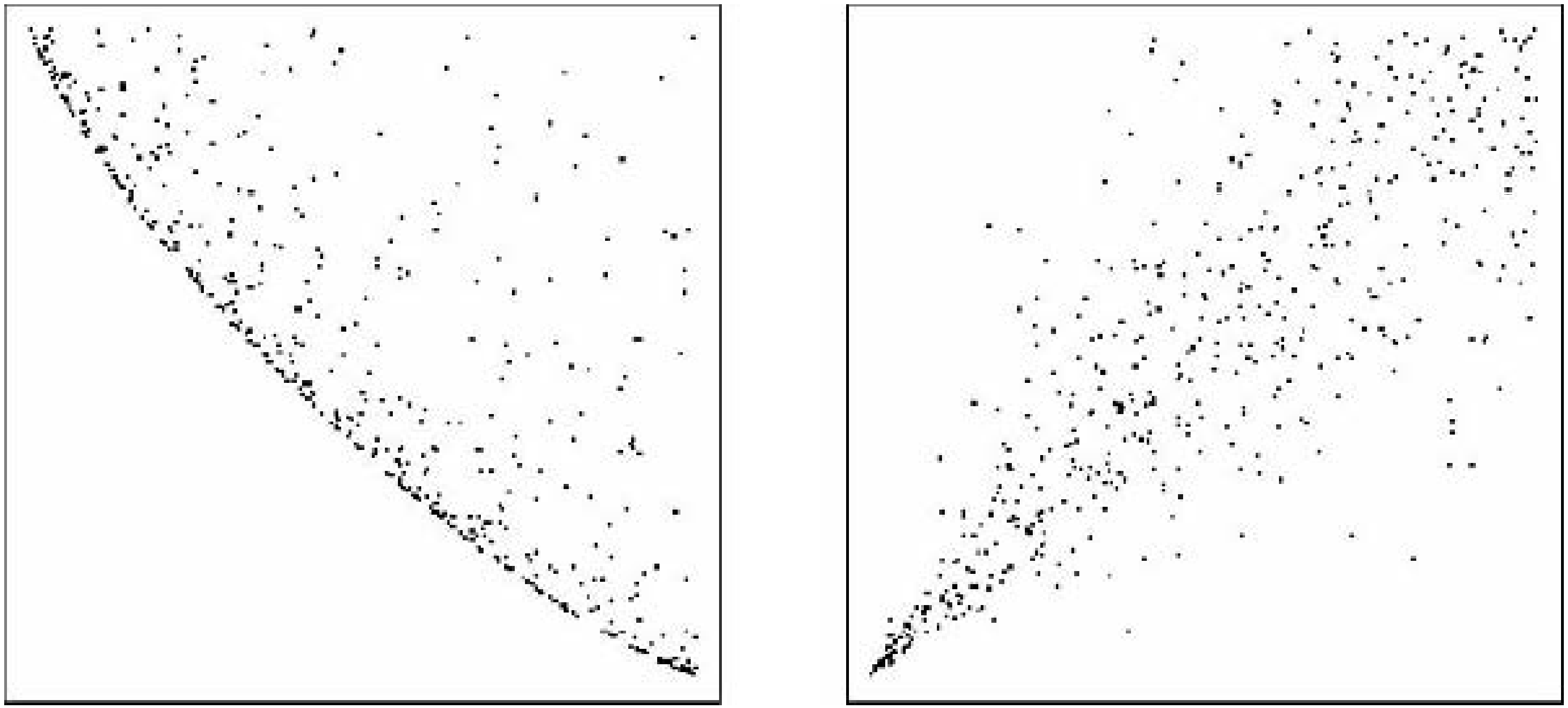


Fig. 4.2. Scatterplots for copulas (4.2.1), $\theta = -0.8$ (left) and $\theta = 4$ (right)

Archimedean Copula Families

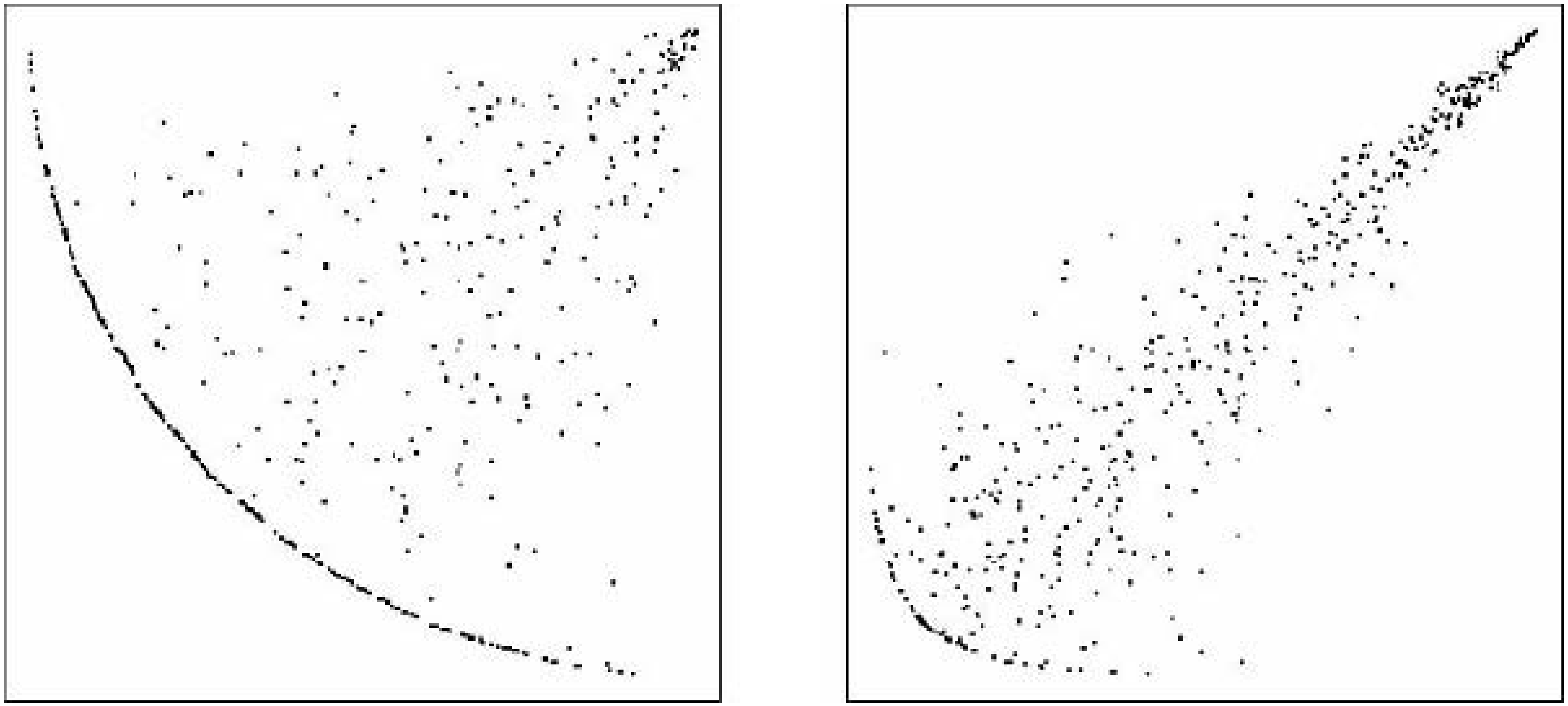


Fig. 4.3. Scatterplots for copulas (4.2.2), $\theta = 2$ (left) and $\theta = 8$ (right)

Archimedean Copula Families

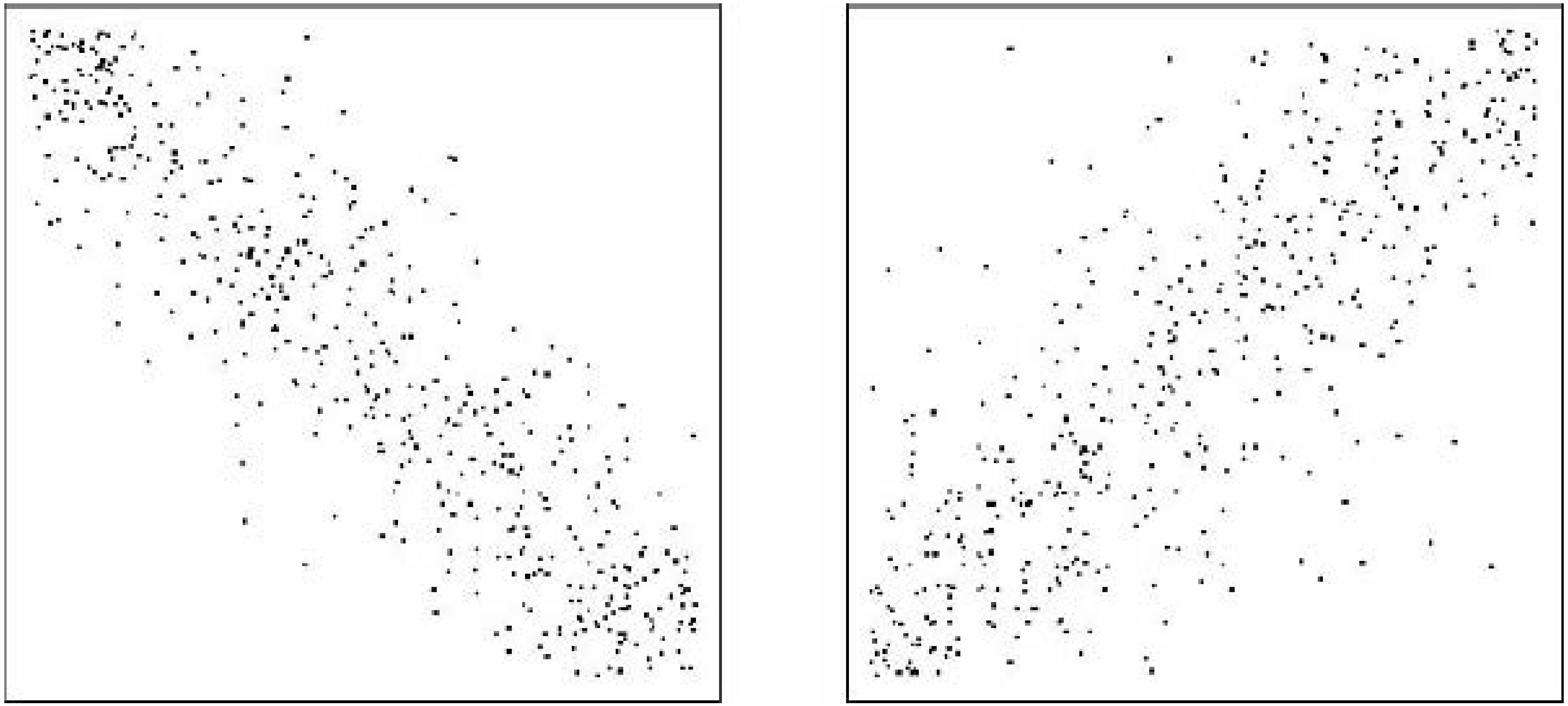


Fig. 4.4. Scatterplots for copulas (4.2.5), $\theta = -12$ (left) and $\theta = 8$ (right)

Archimedean Copula Families

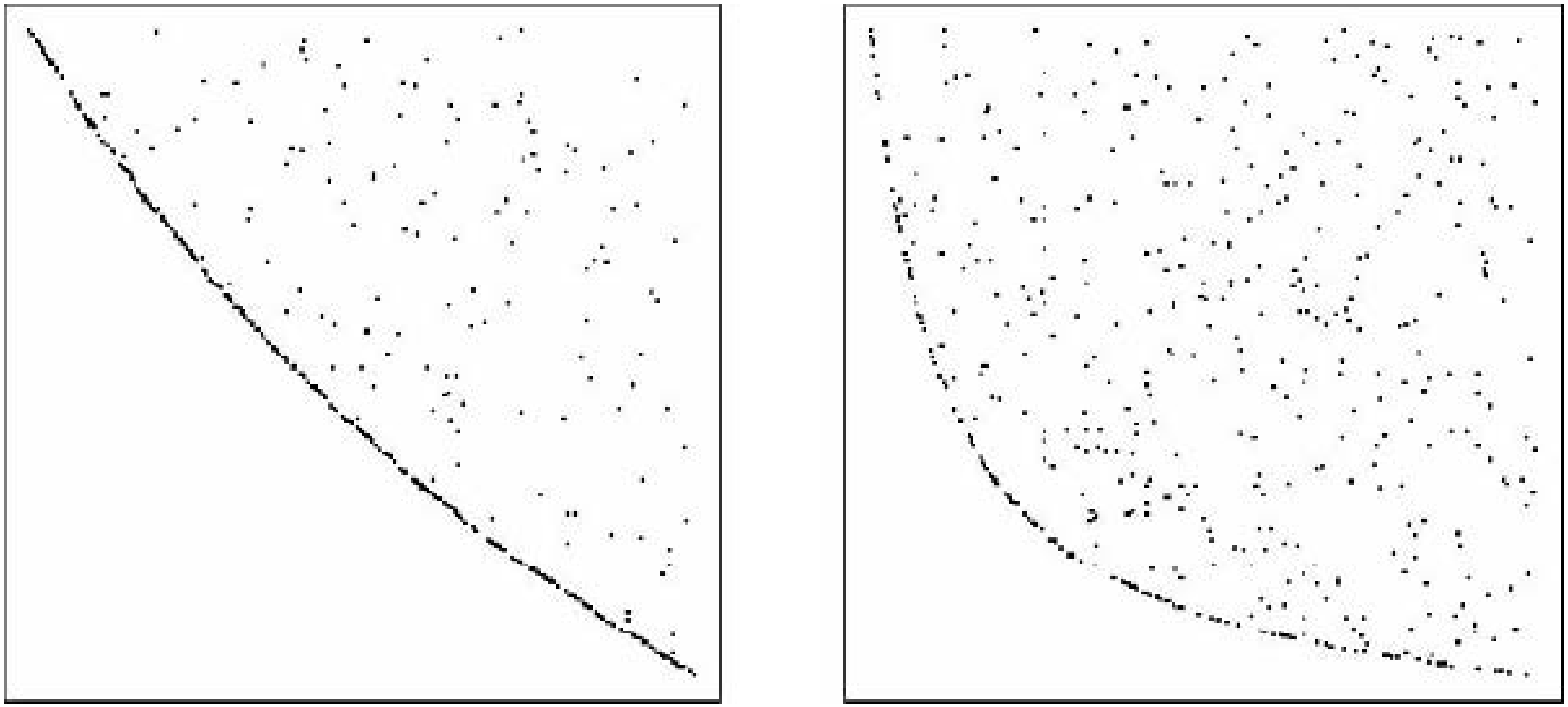


Fig. 4.5. Scatterplots for copulas (4.2.7), $\theta = 0.4$ (left) and $\theta = 0.9$ (right)

Archimedean Copula Families

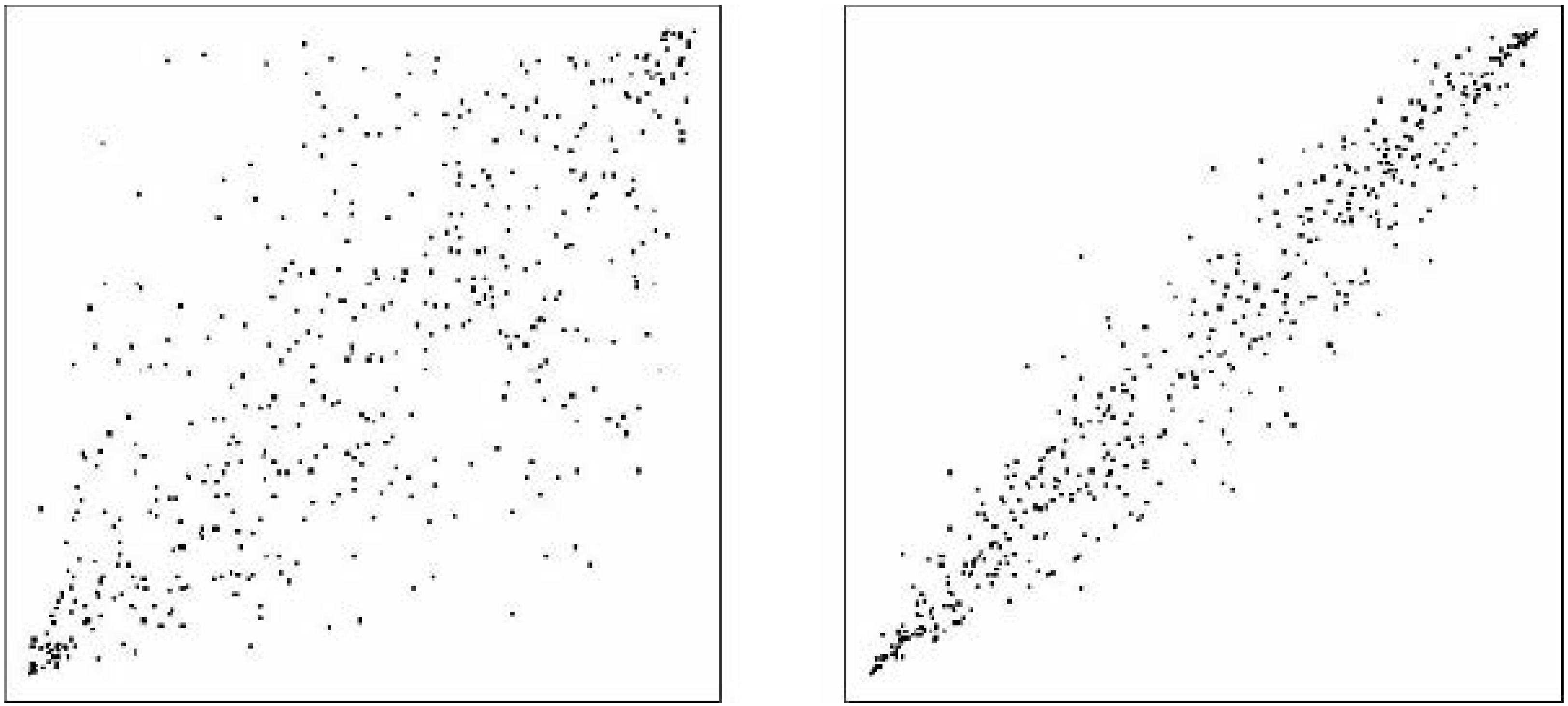


Fig. 4.6. Scatterplots for copulas (4.2.12), $\theta = 1.5$ (left) and $\theta = 4$ (right)

Archimedean Copula Families

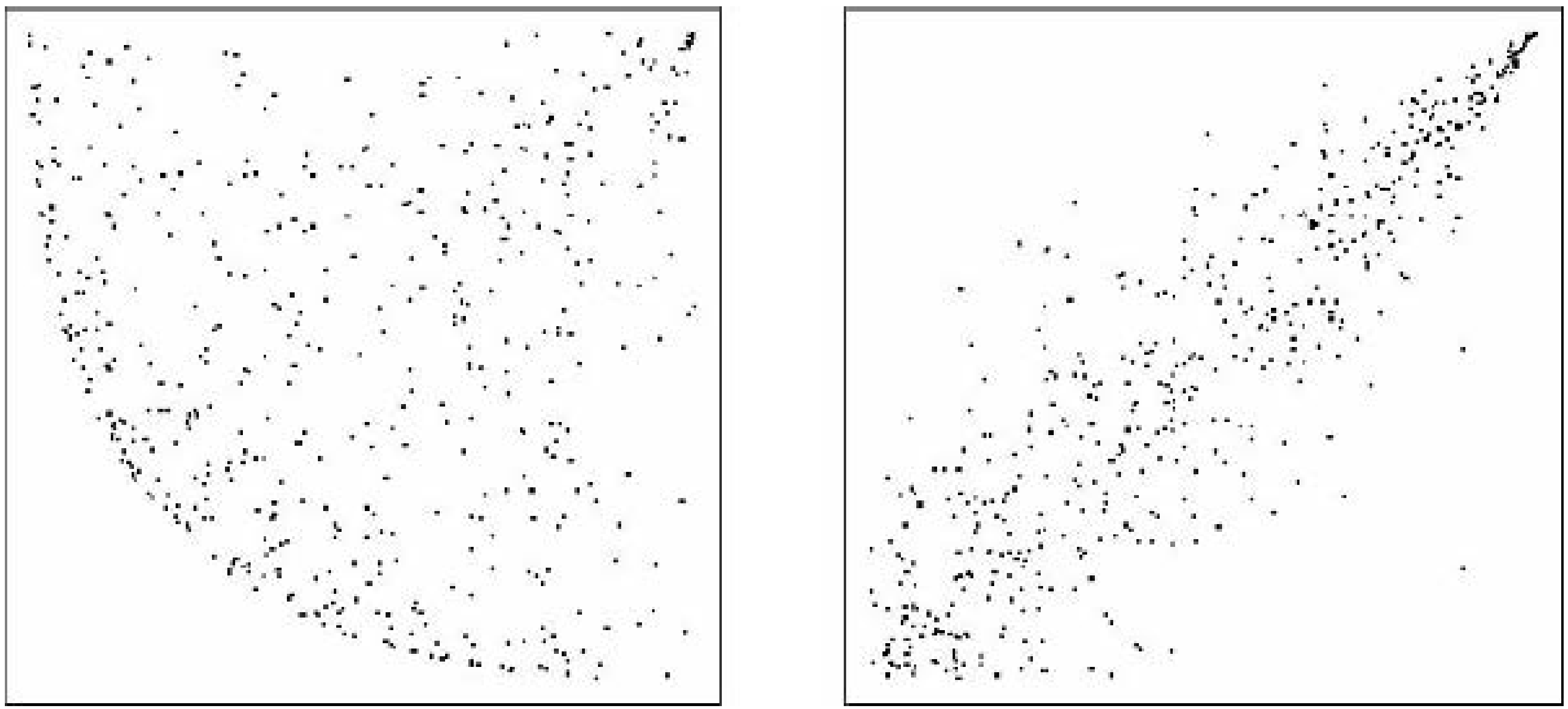


Fig. 4.7. Scatterplots for copulas (4.2.15), $\theta = 1.5$ (left) and $\theta = 4$ (right)

Archimedean Copula Families

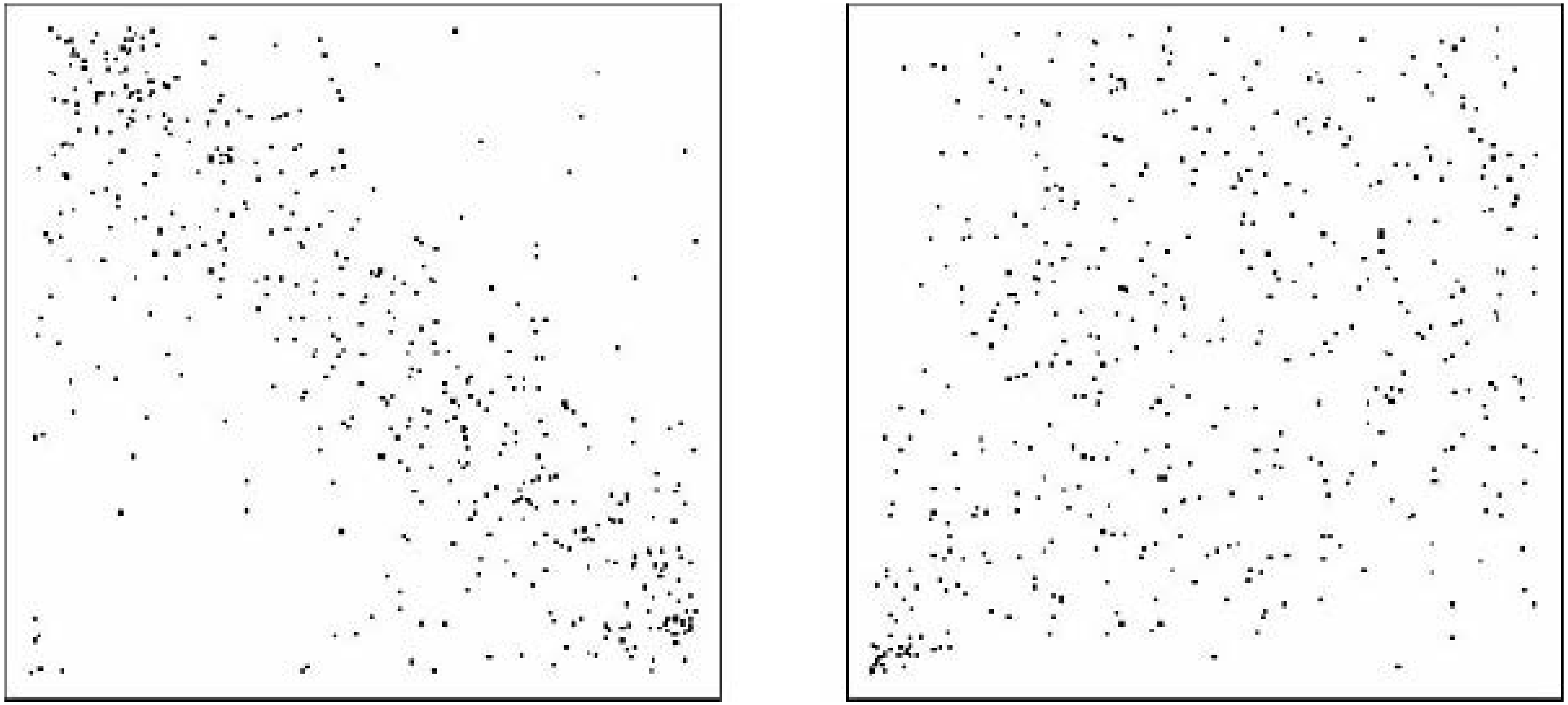


Fig. 4.8. Scatterplots for copulas (4.2.16), $\theta = 0.01$ (left) and $\theta = 1$ (right)

Archimedean Copula Families

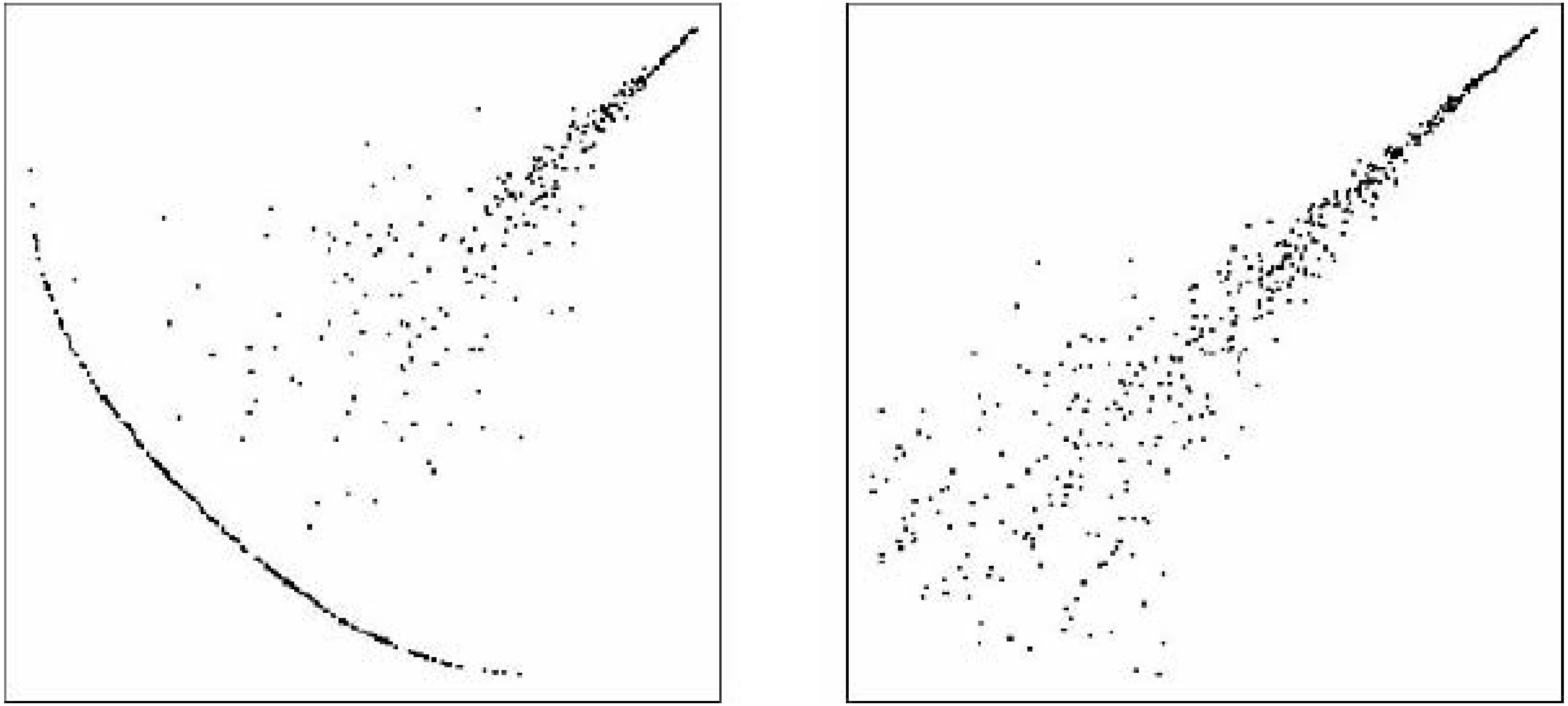


Fig. 4.9. Scatterplots for copulas (4.2.18), $\theta = 2$ (left) and $\theta = 6$ (right)

Copula Constructing

- Multivariate Copula?
- Multiparameter Copula?

Multivariate Archimedean Copulas

Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty)$ such that $\varphi(0)=\infty$ and $\varphi(1)=0$, and let φ^{-1} be the inverse of φ. Then

$$C^n(\mathrm{u}) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2)+\ldots+\varphi(u_n))$$

is a n-copula iff φ^{-1} is completely monotonic on $[0,\infty)$, i.e.

$$(-1)^k \frac{d^k}{dt^k}\varphi^{-1}(t) \geq 0 \text{ for all } t \in \mathrm{int}([0,\infty)) \text{ and } k = 0,1,2,\ldots$$

Multivariate Archimedean Copulas

- Clayton Family

$$C_\theta^n(\mathbf{u}) = \left(u_1^{-\theta} + u_2^{-\theta} + \cdots + u_n^{-\theta} - n + 1\right)^{-1/\theta} \quad \varphi_\theta(t) = t^{-\theta} - 1 \text{ for } \theta > 0$$

- Frank Family

$$C_\theta^n(\mathbf{u}) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)\cdots(e^{-\theta u_n} - 1)}{(e^{-\theta} - 1)^{n-1}}\right)$$

$$\varphi_\theta(t) = -\ln\left((e^{-\theta t} - 1)\big/(e^{-\theta} - 1)\right) \text{ for } \theta > 0$$

Multivariate Archimedean Copulas

- Gumbel-Hougaard Family

$$C_\theta^n(\mathbf{u}) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta + \cdots + (-\ln u_n)^\theta\right]^{1/\theta}\right)$$

$$\varphi_\theta(t) = (-\ln t)^\theta,\ \theta \geq 1$$

- A 2-parameter Multivariate Copula

$$C_{\alpha,\beta}^n(\mathbf{u}) = \left\{\left[(u_1^{-\alpha} - 1)^\beta + (u_2^{-\alpha} - 1)^\beta + \cdots + (u_n^{-\alpha} - 1)^\beta\right]^{1/\beta} + 1\right\}^{-1/\alpha}$$

$$\varphi_{\alpha,\beta}(t) = (t^{-\alpha} - 1)^\beta \text{ for } \alpha > 0,\ \beta \geq 1$$

Estimating Copula Parameters

- Copula Density

$$c^n(\mathrm{u}) \equiv \frac{\partial^n}{\partial u_1 \ldots \partial u_n} C^n(\mathrm{u}) \text{ if it exists on } \mathrm{int}(\mathrm{I}^n)$$

- Density

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} H(\mathrm{x}) = c^n(F_1(x_1), \ldots, F_n(x_n)) f_1(x_1) \ldots f_n(x_n)$$

Estimating Copula Parameters

- Log-likelihood

$$\log c^n(F_1(x_1;\alpha_1),\ldots,F_n(x_n;\alpha_n);\theta) + \sum_i \log f_i(x_i;\alpha_i)$$

- Two-step Estimation: Plug-in MLE

  Genest et al. (1995). *Biometrika*

- One-step Efficient Estimation: Sieve MLE

  Chen et al. (2006). *J.A.S.A*

Copula Selection

- Goodness-of-fit Tests.

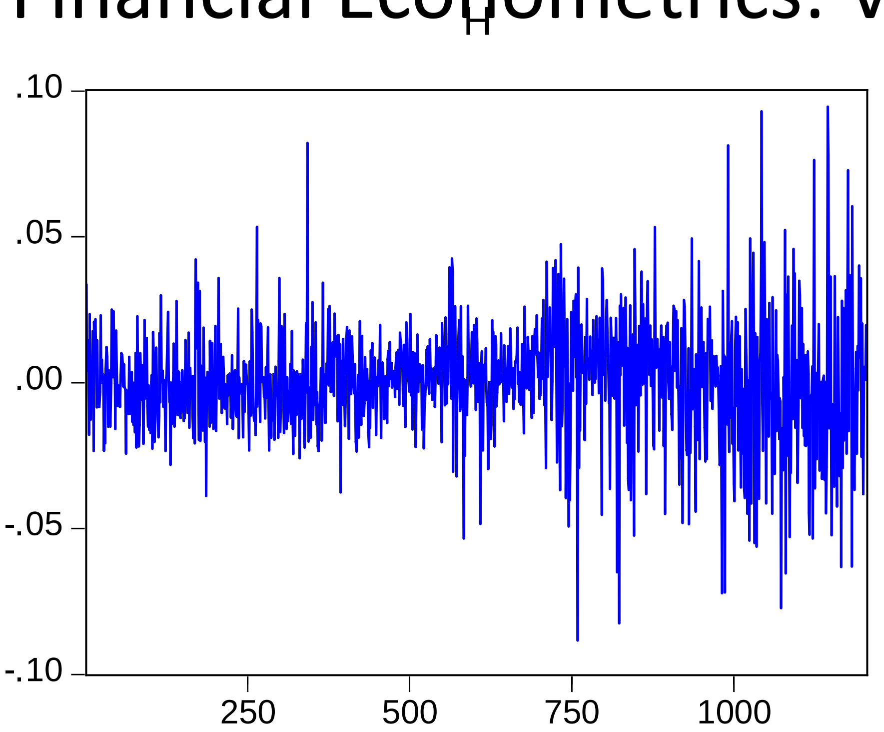
  Chen et al. (2005). *Canadian Journal of Statistics*

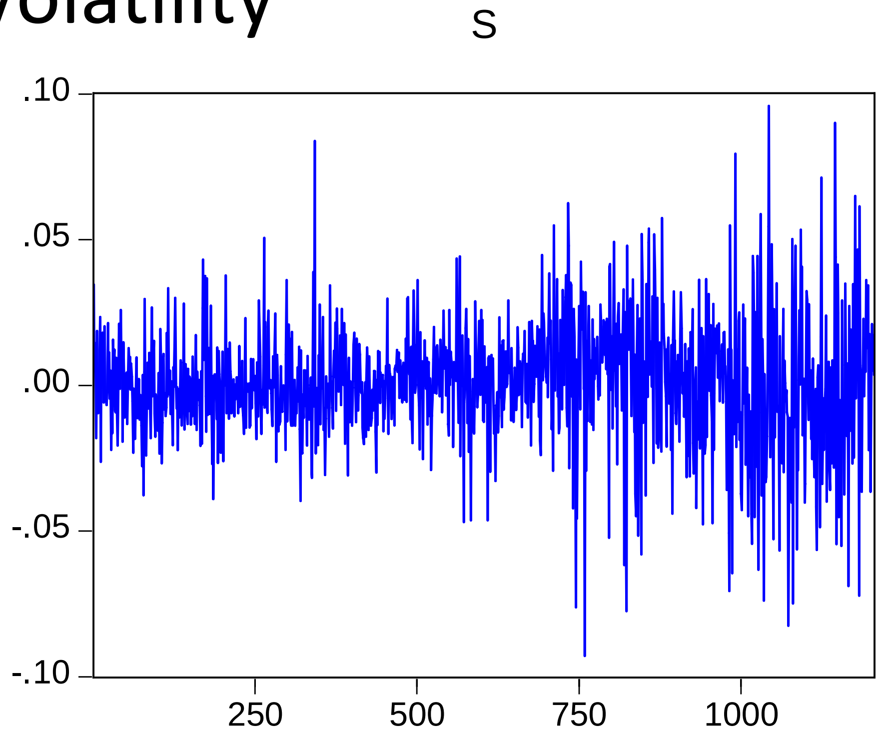
  Chen et al. (2006). *Journal of Econometrics*

  Genest et al. (2009). *Insurance: Mathematics and Economics*

An Application Case

Financial Econometrics: Volatility





2004.1.5-2008.12.19

An Application Case

Model:

Multivariate Generalized Autoregressive Conditional Heteroskedasticity (MGARCH)

Bauwens et al. (2006). *Journal of Applied Econometrics*

$$r_t = H_t^{1/2} z_t$$

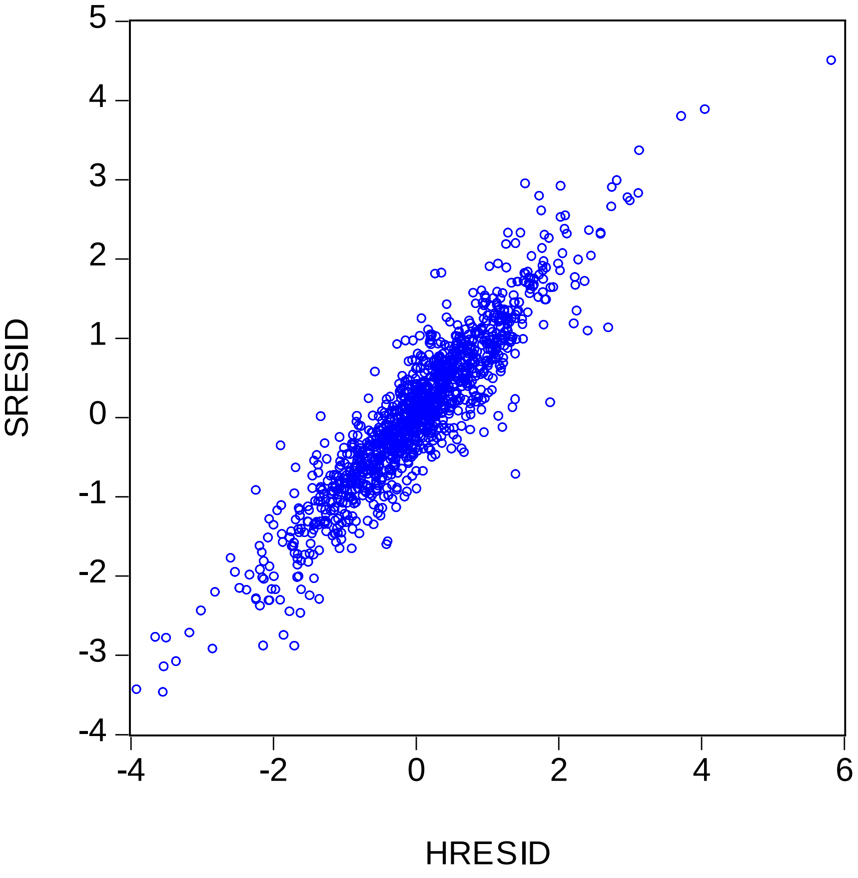
An Application Case

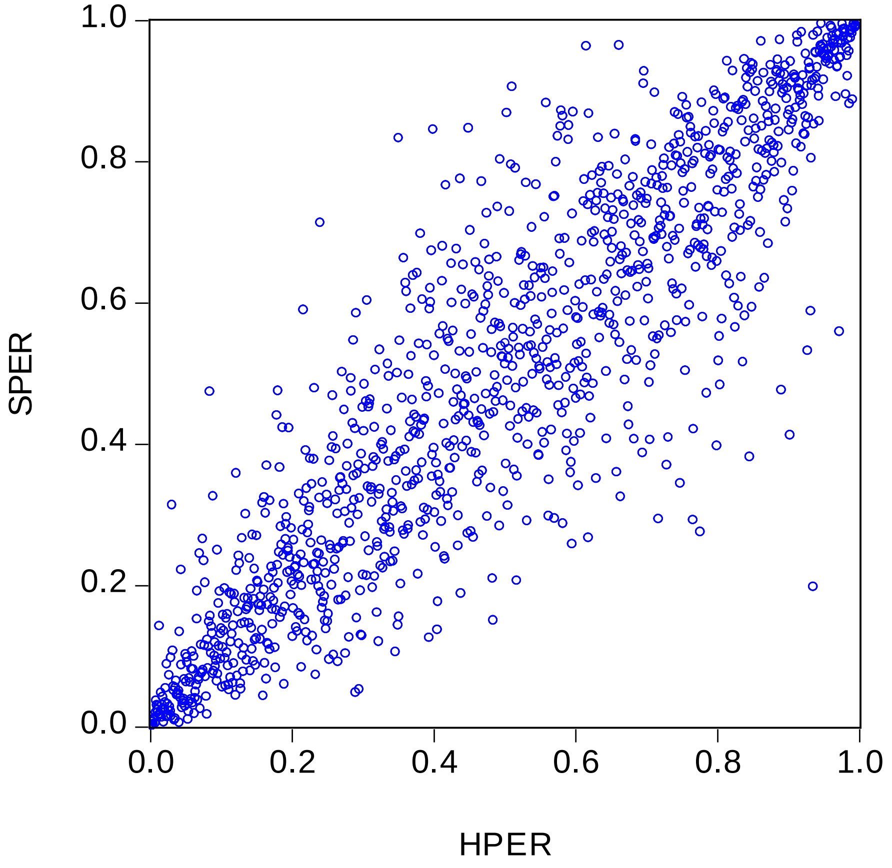
Copula-MGARCH

$$r_{it} = h_{it}^{1/2} z_{it} \quad i = 1, \ldots, n$$

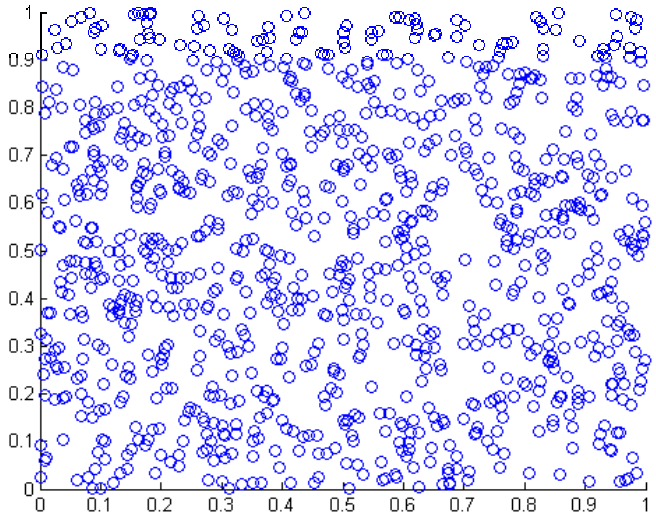
$$z_t = (z_{1t}, \ldots, z_{nt}) \sim C(F_1(z_{1t}; \theta_1), \ldots, F_n(z_{nt}; \theta_n); \alpha)$$

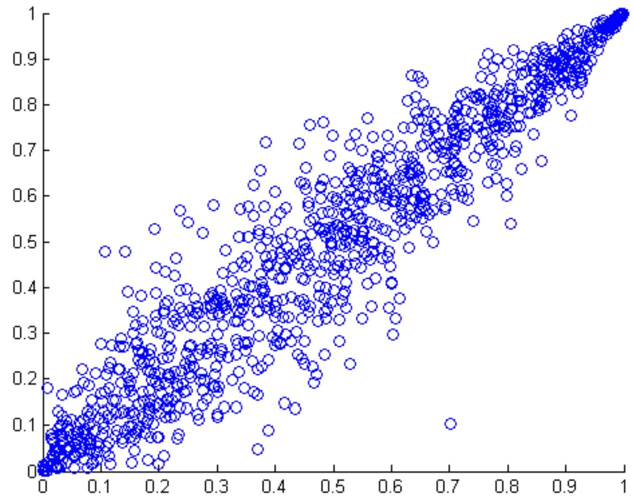
An Application Case

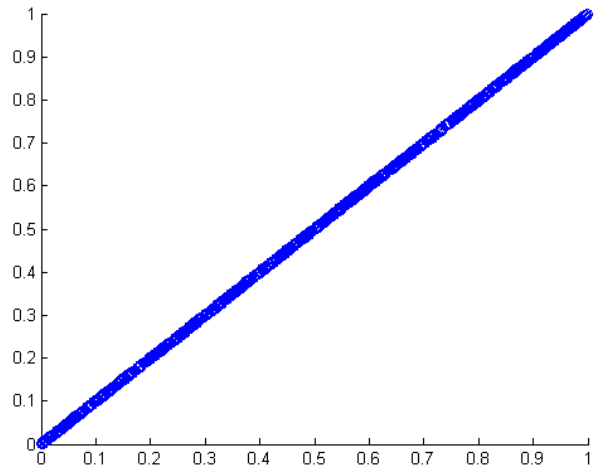




An Application Case

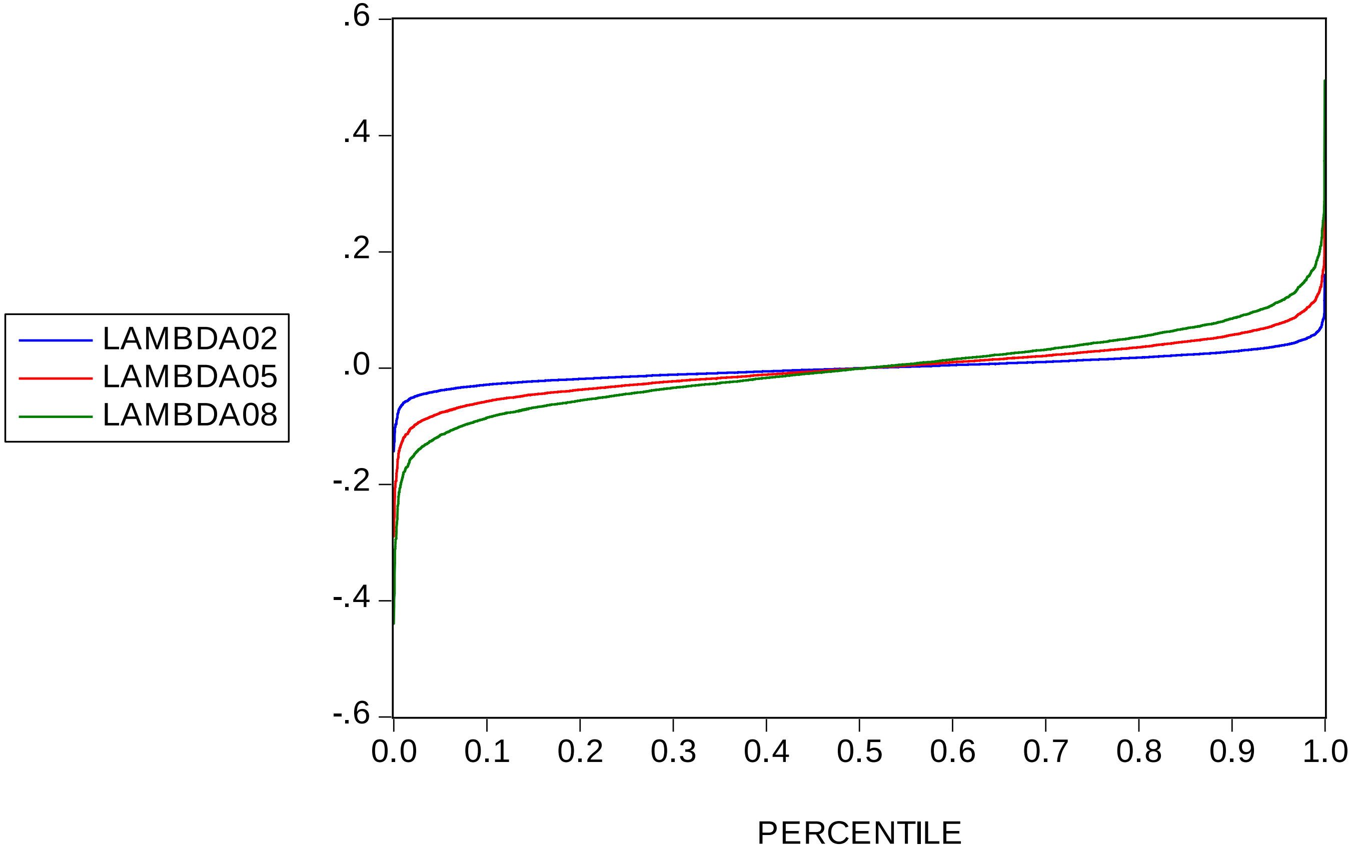






Gumbel Copula (Parameter=1,5,1000)

An Application Case



$$\lambda r_{ht} + (1-\lambda) r_{st} \quad \lambda \in (0,1)$$

Monographs about Copula

- Cherubini U., Luciano E., Vecchiato, W. (2004). Copula Methods in Finance. Wiley
- Nelson, Roger B. (2006). An Introduction to Copulas. Springer